Redesign of nonlinear Duffing oscillators to cause linear-like behaviour: two approaches

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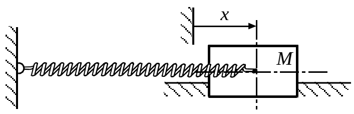
Abstract. Linear oscillators are known to have a constant, amplitude-independent frequency. On the other side, classical Duffing oscillators, which differ from them only by the existence of a cubic geometric term in the equation of motion, have the frequency that changes with their amplitude. In this study, two approaches are presented on how to redesign Duffing oscillators and cause linear-like behaviour with an amplitude-independent frequency. The first approach is related to the modification of some internal characteristics through the kinetic energy. The second approach involves the use of a specially designed multi-term external excitation. The analytical results are verified numerically and two illustrative examples are given.

Keywords: linear oscillators; Duffing oscillators; frequency; amplitude; external excitation.

1 Introduction

A Simple Harmonic (linear) Oscillator (SHO) and the Duffing nonlinear Oscillator (DO) are the paradigms of Linear Vibration Theory and Nonlinear Dynamics, respectively. Given the fact that they model a variety of different physical and engineering systems, their free and forced responses have been investigated in details [1-3]. The main characteristics of these two types of oscillators are summarised in Table 1, including the associated type of restoring force, equations of motion, solutions for motion and the period. Note that only the Hardening DO (HDO) is considered herein: it differs from the SHO, which is characterised by the positive stiffness k, by the additional cubic term in the restoring force and in the equation of motion (its both stiffness coefficients are positive, i.e. k_1, k_3>0). As a result of this, the exact solutions for free vibrations of the SHO and the HDO are given in different forms: for the former, they are expressed in terms of a harmonic function, and for the latter in terms of the Jacobi cn elliptic function. By using a Fourier series expansion, the Jacobi cn elliptic function can be expressed as a sum of odd harmonics: it includes a fundamental harmonic and higher harmonics as well. The additional difference between these two types of oscillators is related to their frequency/period of free vibrations: for the SHO, they are constant (amplitude-independent), while for the HDO, they are amplitude-dependent. This leads us to the question of redesigning the HDO so that its restoring force stays the same, but changing its kinetic energy or adding certain external excitation to yield an amplitude-independent period, as in the SHO. This is called herein 'linear-like' behaviour as it is concerned both with autonomous and non-autonomous oscillators. In the literature, a fixed, amplitude-independent period is related to the so-called property of isochronicity [4, 5].

Table 1. Characteristics of the SHO and the HDO and their free response

Model		
Type	SHO	HDO
Restoring (spring) force	Linear function of the displacement	Linear-plus-cubic function of the displacement
Equation of motion	$M\ddot{x}+kx=0$	$M\ddot{x}+k_1x+k_3x^3=0$
Solution for motion	$x(t)=A\cos\omega t$ Frequency: $\omega=\sqrt{\frac{k}{M}}$	$x(t)=A\mathrm{cn}(\omega t\vert m)$ Frequency: $\omega=\sqrt{\frac{k_1+k_3A^2}{M}}$ Elliptic modulus: $m=\frac{k_3A^2}{2(k_1+k_3A^2)}$
Period	Constant, amplitude-independent $T=\frac{2\pi}{\omega}$	Amplitude-dependent $T=\frac{4K(m)}{\omega}$ K - complete elliptic integral of the first kind

2 First approach: Change of internal characteristics

The first redesign of the HDO relates its internal characteristics and is based on the transformation approach [6, 7], in which the kinetic energy E_k and potential energy E_p of nonlinear oscillators are made equal to the one of the SHO:

$$E_{k\mathrm{SHO}} = \frac{1}{2}\dot{x}^2, \quad E_{p\mathrm{SHO}} = \frac{1}{2}x^2. \tag{1}$$

Note that the masses are assumed to be equal to the unit mass, but in case this does not hold, an appropriate normalization can be carried out to make the inertial coefficients equal to unity. Based on the equality of the potential energy $E_p \equiv E_{p\mathrm{SHO}} = x^2/2$, one has

$$x = \sqrt{2E_p}. \tag{2}$$

Then, the kinetic energy E_k of nonlinear oscillators is also made equal to the one of the SHO. Using Eq. (2), one can derive

$$E_k = \frac{\dot{x}^2}{2} = \frac{\left(E_p'\right)^2}{4E_p}\dot{x}^2, \tag{3}$$

where $E_p' = dE_p/dx$. Lagrange's equation of the second kind yields the following equation of motion

$$\ddot{x} + \left(\frac{E_p''}{E_p'} - \frac{E_p'}{2E_p}\right)\dot{x}^2 + \frac{2E_p}{E_p'} = 0. \tag{4}$$

The last term on the left-hand side $2E_p/E_p'$ is required to correspond to the HDO force $F = x + x^3$, which gives the potential energy $E_p = x^2/\left(2\left(1+x^2\right)\right)$. Equation (3) now gives the kinetic energy $E_k = \dot{x}^2/\left(2\left(1+x^2\right)^3\right)$, so that Eq. (4) becomes

$$\ddot{x} - \frac{3x}{1+x^2}\dot{x}^2 + x + x^3 = 0. \tag{5}$$

So, this is the modified equation of motion of the HDO which will have the constant, amplitude-independent period as the SHO. The redesign obviously resulted in the term $-3x\dot{x}^2/\left(1+x^2\right)$, which stems from the displacement-dependent kinetic energy.

In addition, knowing that the SHO has the energy conservation law in the form $x^2/2+\dot{x}^2/2=const.$, and using Eqs. (2), (3) and (6), the following first integral for the redesigned HDO is obtained

$$\frac{1}{\left(1+x^2\right)^3}\dot{x}^2+\frac{x^2}{1+x^2}=\frac{A^2}{1+A^2}. \tag{6}$$

Using a harmonic solution for the free response of the SHO (see Table 1) as well as Eqs. (2) and (5), the implicit expression for the solution for motion of the redesigned HDO is derived

$$\frac{A}{\sqrt{1+A^2}}\cos t=\frac{x}{\sqrt{1+x^2}}. \tag{7}$$

2.1 Example 1

To illustrate the behaviour of the redesigned HDO, a numerically obtained time response from Eq. (5) is plotted in **Fig. 1a** in the red dots for $x(0)=1$ and zero initial velocity, while the analytical solution given by Eq. (7) with $A=1/\sqrt{2}$ is depicted by the black solid line. **Fig. 1a** includes the analytical and numerical solutions shown for two additional pairs of the initial conditions $x(0)=0.1;0.5$ and zero initial velocity. All these time histories have the same, constant period. This confirms that the period/frequency is amplitude-independent. Phase trajectories obtained numerically from Eq. (5) and based on Eq. (6) are plotted in **Fig. 1b**. Both **Fig. 1a** and **Fig. 1b** validate the analytical results derived.

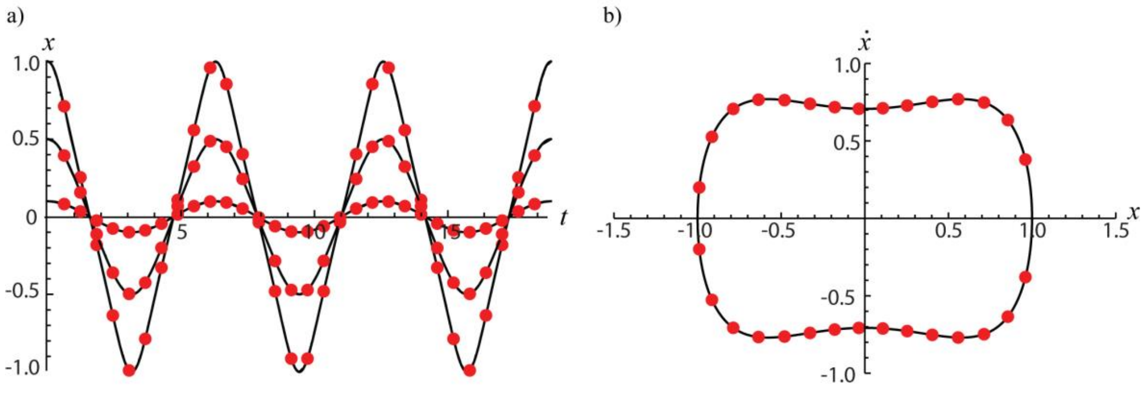


Fig. 1. a) Time response: b) Phase trajectories. Numerically obtained solution from Eq. (5) - red dots: analytical solution, Eq. (7) - black solid line.

3 Second approach: Use of external excitation

To redesign the HDO in a different way, the external force is introduced to act on it, so that its equation of motion becomes

$$\ddot{x}+k_1x+k_3x^3 = f(t). \tag{8}$$

To find the time-dependent form of the external force, it is assumed as $f(x)=Bx+Dx^3$, with x=x(t). Now, Eq. (8) transforms into

$$\ddot{x}+(k_1-B)x+(k_3-D)x^3=0. \tag{9}$$

This system will behave as the SHO provided that $D=k_3$ and $k_1>B$. The response of the resulting SHO is then given by

$$x_r = A\cos(\omega_r t), \tag{10}$$

where

$$\omega_r=\sqrt{k_1-B}. \tag{11}$$

The parameters existing in this approach are: A, B, $D=k_3$ and ω_r. However, there is only one relationship between them, Eq. (11). Thus, D is defined by $D=k_3$, two more parameters can be chosen in advance (one of which is A) and one is calculated consequently.

Substituting Eq. (10) into $f(x)=Bx+Dx^3$ and then into Eq. (8), the equation of motion of the externally excited HDO that will behave as the free vibrating SHO is obtained:

$$\ddot{x}+k_1x+k_3x^3=\left(BA+\frac{3}{4}A^3k_3\right)\cos\left(\sqrt{k_1-B}\,t\right)+\frac{1}{4}A^3k_3\cos\left(3\sqrt{k_1-B}\,t\right). \tag{12}$$

3.1 Example 2

To illustrate this behaviour, the parameters of the HDO are defined to be k_1=1, k_3=1. Further, it is assumed that A=1, B=1/2. The frequency of the response given by Eq.

(11) is then calculated to be $\omega_r = \sqrt{1/2}$. **Fig. 2** shows the numerical solutions of Eq. (12) depicted by the black solid line, while the analytical response (10), (11) is depicted by the red dots. These two solutions coincide. To illustrate the change of the response, the numerical solution of the original equation of motion of the HDO, Eq. (8) with $f(t)$=0, is plotted as the green dotted line. Two different values of the amplitude are used (A=1 in **Fig. 2a** and A=1/2 in **Fig. 2b**), and it is seen that the period is amplitude-dependent in the original HDO, while it stays constant in the externally excited HDO (the resulting SHO).

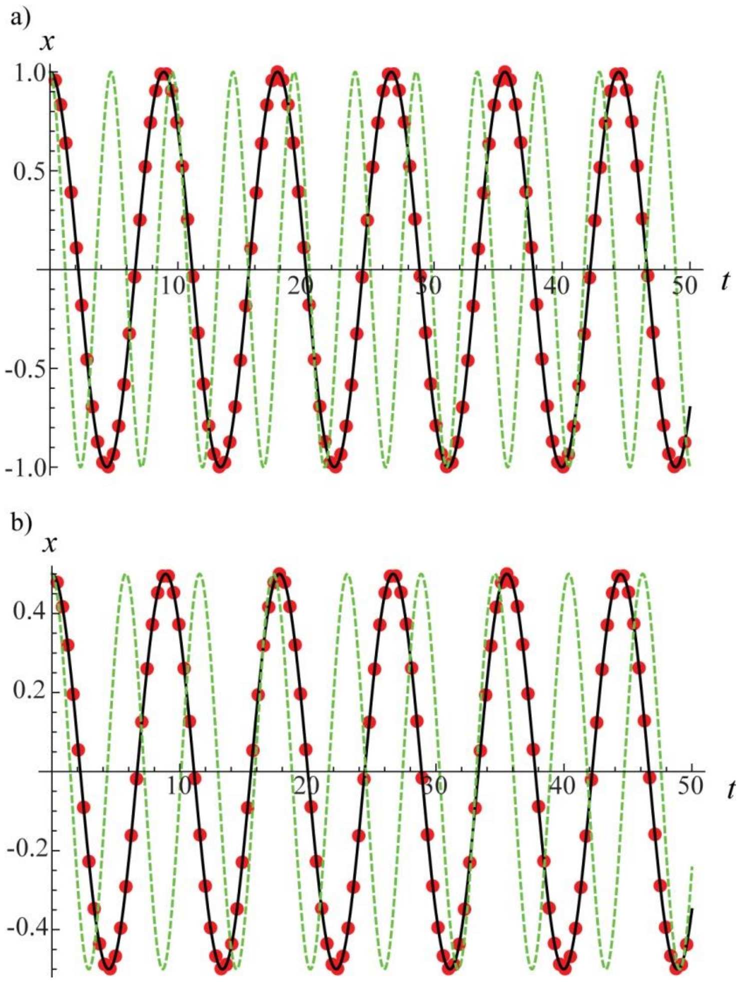


Fig. 2. Time responses corresponding to k_1=1, k_3=1, B = 1/2, D= k_3, and: a) A=1, b) A =1/2. Numerical solution of Eq. (12) - black solid line; analytical response Eqs. (10), (11) - red dots; numerical solution of Eq. (8) with $f(t)$=0 - green dotted line.

4 Conclusions

In this work, nonlinear oscillators with a hardening Duffing restoring force have been considered. Unlike conservative Simple Harmonic Oscillators, which have an amplitude-independent period, the conservative Hardening Duffing Oscillators have the period that does change with their amplitude and are, thus, non-isochronous. Two approaches have been presented on how to redesign Hardening Duffing Oscillators by keeping their restoring force in the same linear-plus-cubic form to make them have the period that does not change with their amplitude. The first approach involves making their kinetic energy be of a specific, displacement-dependent form. The second approach encompasses adding a specially designed multi-term external excitation. Numerical verifications of the isochronous behaviour have been provided.

Although these two approaches have been presented for Hardening Duffing Oscillators only, they have wider applications and can further be extended to other types of nonlinear oscillators.

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